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DYNAMICS OF A GAS CONTAINING SMALL SOLID PARTICLES

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It is shown that for gas-particle flow systems, the particle flow field and the mutual interaction of gas with the particle cloud are governed by four similarity parameters, each of which has a simple physical significance. (1) The velocity equilibration parameter, k_v/L : the ratio of the distance required for particle velocity to reach that of the fluid, to the characteristic geometric length of the flow field. (2) The thermal equilibration parameter, λ_T/L : the ratio of distance required for particle temperature to reach that of the fluid, to the characteristic geometric length of the flow field. (3) The momentum interaction parameter, κ : the ratio of particle mass density to fluid mass density, which governs the relative acceleration experienced by fluid and particle cloud as a result of interaction. (4) The thermal interaction parameter, $\kappa(c_s/c_p)$: the ratio of thermal capacity of solid phase with that of the gas, which is a measure of relative temperature changes during equilibration. These parametric groups must be augmented when the particles collide or when they radiate to each other. General equations for gas-particle flow systems are derived on the basis of a particle distribution function. When the problem involves particles of only one size, these relations simplify accordingly.

Using the normal shock wave and the one-dimensional gas-dynamics as examples, it is shown that the approximations $\lambda_v/L \gg 1$ and $\lambda_v/L \ll 1$ respectively can be used to advantage. The channel flow problem is solved in general for "small slip" approximation $\lambda_v/L \ll 1$.

The trajectories of single particles in Prandtl-Meyer flow are obtained as an illustration of the technique appropriate to the approximation $\kappa \ll 1$, that is, the particle density is so low that it does not affect significantly the gas motion.

Finally, the Blasius problem is extended to fluid-particle systems utilizing an expansion in powers of λ_v/x . This solution, valid for distances x more than a few λ_v downstream from the leading edge, leads to a new shear law and a new heat transfer coefficient appropriate to the fluid-particle boundary layer.

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NOTATION

x, y, z or x_1, x_2, x_3	Cartesian coordinates
t	time
u, v, w or u_1, u_2, u_3	Cartesian velocity components
m	particle mass
σ	particle radius
μ	fluid viscosity
ρ	mass density
ν	kinematic viscosity of fluid
k	thermal conductivity of fluid
p	fluid pressure
T	fluid temperature
a	speed of sound
r	radial distance from origin to particle location
$\xi = x/L$	dimensionless distance along flow direction
A	cross-sectional area of channel or duct
c_p, c_v	specific heats of gas
$\gamma = c_p/c_v$	specific heat ratio of gas
c_s	specific heat of solid materials
M	gas Mach number
τ	characteristic time—associated with equilibration of particle
λ	characteristic length associated with equilibration of particle
L	characteristic geometric length associated with flow field
n	number of particles per unit volume
l	distance particle is transported by fluid before it encounters another particle
α	Stefan's constant
F_p	force per unit volume exerted by particles on fluid
Q_p	heat transfer rate per unit volume from particles to fluid
c_1, c_2, c_3	peculiar velocity components of single particle
ϑ	peculiar temperature of single particle, also : angle measured from initial Mach line
$f(x_i, c_j, \vartheta, \sigma, t)$	particle distribution function
$L(\vartheta)$	angular momentum of particle about origin
q^*	maximum velocity obtainable for a gas
β	$\sqrt{\left(\frac{\gamma - 1}{\gamma + 1}\right)}$
R_e	particle Reynolds number based upon slip velocity
$\nu^* \equiv \frac{\nu}{1 + \kappa}$	kinematic viscosity of fluid-particle mixture in equilibrium
$\eta = \frac{y}{\sqrt{[(\nu^* x/u_0)]}}$	boundary layer coordinate normal to surface
$\psi(x, \eta)$	boundary layer stream function
$f^{(i)}(\eta)$	coefficient in expansion of stream function

$g^{(i)}(\eta)$	coefficient in expansion of particle velocity
$h^{(i)}(\eta)$	coefficient in expansion of particle mass density

Subscripts

p	particle
s	solid ; also slip
v	velocity ($\lambda_v \rightarrow$ velocity equilibration length)
T	thermal ($\lambda_T \rightarrow$ thermal equilibration length)
0	reference state ; free stream
R	radiation ($l_R \rightarrow$ radiation mean free path)
—	bar used to indicate values of quantities when fluid and particles are in velocity and thermal equilibrium

PARAMETRIC GROUPS

$\frac{\tau_v}{\tau} ; \frac{\lambda_v}{L}$	particle slip parameter
$\frac{\tau_T}{\tau} ; \frac{\lambda_T}{L}$	particle temperature lag parameter
$\kappa = \rho_p / \rho$	particle density ratio at reference state
c_s / c_p	solid-fluid specific heat ratio
$\frac{\lambda_m}{L} \equiv \kappa \left(\frac{\sigma u_s}{\nu} \right) \left \frac{\sigma_2 - \sigma_1}{\sigma} \right $	particle collision parameter
$\frac{\alpha T^4}{\rho_p u_0 c_s T} \left(\frac{l_R}{L} \right)$	radiation parameter ; flow direction
$\frac{k_R}{k}$	radiation parameter ; normal to flow direction
$\frac{k_R}{k} \left \frac{\sigma_2 - \sigma_1}{\sigma} \right \frac{\sigma}{l_R}$	radiation parameter ; particle-particle exchange

1. INTRODUCTION

The transport of solid particles by fluids has, from historic times, been the object of scientific and engineering research. The problem has appeared in various forms such as sediment transport by water and by air¹, the collection of ice on buildings and aircraft structures^{2,3}, the centrifugal separation of particulate matter from fluids, fluid-droplet sprays^{4,5}, fluidized beds and other two-phase flow phenomena of interest in chemical processing⁶, and the electrostatic precipitation of dust. Recently, however, the motion of solid particles in rocket motor exhaust^{7,8,9} has focused attention upon some new features of the problem and has increased scientific interest in the entire field. Much of the significant literature bearing upon the rocket nozzle problem has been reviewed by Hoglund¹⁰.

To the present time, however, the dynamics of fluid-particle systems has remained closely associated with particular detailed problems and has not found a place in the general discipline of fluid mechanics. This is in

no small part due to inherent complexity and unique difficulties of each example and to the inadequacy of many standard analytical techniques to deal with them.

It is the aim of this paper to show how fluid-particle flow phenomena fit into and extend the pattern of fluid mechanics research. First, the significant new dimensionless parameters are introduced and their general physical effect on the solution is indicated. Second, general statements are developed of the dynamic and thermodynamic relations that govern fluid-particle motion. Finally, the analytical solutions of several typical problems will be summarized as an illustration of the effects of the new dimensionless parameters, as an indication of the analytical methods appropriate to various problems and to point out some areas in which the field of fluid-particle dynamics may profitably be advanced.

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2. BASIC PARAMETERS AFFECTING PARTICLE MOTION

The motion of a gas containing solid particles is critically dependent upon the interaction between fluid and particles. In general, the force of interaction depends upon the particle Reynolds number, Mach number, Knudsen number, as well as details of the local flow field and the interactions between particles. Here we shall assume the Reynolds number and the molecular mean free path to be small enough that Stokes drag law is a reasonable approximation and that the material constituting the particles is dense enough that their volumetric effect is negligible. These assumptions are largely a matter of convenience; they permit criteria and results to be made definite and relatively simple. Most of the considerations here may be carried over, with appropriate numerical effort, to more complex interaction laws.

Fluid-Particle Interaction

Consider a single particle of mass m and velocity u_p moving parallel with a gas of velocity u . If the force accelerating the particle toward the fluid speed is given by Stokes' law, the process takes place within the characteristic time $m/6\pi\sigma\mu$. We shall denote

$$\tau_v \equiv \frac{m}{6\pi\sigma\mu} = \frac{2}{9} \left(\frac{\rho_s}{\rho} \right) \left(\frac{\sigma^2}{\nu} \right) \quad (2.1)$$

the velocity equilibration time of the particle. The ratio ρ_s/ρ , of the mass density of the solid constituting the particles to the mass density of the fluid, has been introduced; for metallic solids and gases at standard

conditions, ρ_s/ρ is of the order 10^3 . Correspondingly, if u_0 is a characteristic velocity of the system, the *velocity range* of the particle is

$$\lambda_v \equiv \tau_v u_0 = \frac{2}{9} \left(\frac{\rho_s}{\rho} \right) \left(\frac{\sigma^2 u_0}{\nu} \right) \quad (2.2)$$

Physically λ_v and τ_v are respectively the distance covered and time elapsed for a particle to reduce its relative velocity to e^{-1} of its original value.

Similar considerations hold for the particle temperature T_p which may be out of equilibrium with the temperature T of the fluid. When Stokes drag law is a reasonable approximation, the Nusselt number based on the particle radius is unity. If c_s is the specific heat of the solid and k the thermal conductivity of the gas, the equilibrium time and *thermal range* are

$$\tau_T \equiv \frac{1}{3} \text{Pr} \left(\frac{c_s}{c_p} \right) \left(\frac{\rho_s}{\rho} \right) \frac{\sigma^2}{\nu} \quad (2.3)$$

and

$$\lambda_T \equiv \frac{1}{3} \text{Pr} \left(\frac{c_s}{c_p} \right) \frac{\rho_s}{\rho} \frac{\sigma^2 u_0}{\nu} \quad (2.4)$$

where $\text{Pr} \equiv c_p \mu / k$ is the fluid Prandtl number and c_s is its specific heat at constant pressure. For most gases and for metal particles, the term $\frac{1}{3} \text{Pr} (c_s/c_p)$ is about $\frac{2}{9}$ and consequently the thermal and velocity ranges are effectively equal.

Now if L is the characteristic geometric dimension of the entire flow system, the situation $\lambda_v \gg L$ indicates that the particle enters and leaves the region of interest before there is opportunity to alter its state appreciably. Hence for $\lambda_v/L \gg 1$ or $\tau_v/\tau \gg 1$, the particle motion depends largely upon its initial conditions, that is by its state at the time it enters the system. On the other hand, when $\lambda_v \ll L$, the particle has time to adjust to the local gas motion before it has moved appreciably through this region. If the gas motion were uniform, the particle would achieve this identical uniform motion after travelling a negligible fraction of the region. If the fluid motion is accelerated, the particle adjusts itself to this situation by taking on a slip velocity that provides the force to accelerate the particle at nearly the local rate. The important feature here is that the particle motion depends upon local flow acceleration and is relatively independent of its previous history. Thus for $\lambda_v/L \ll 1$ or $\tau_v/\tau \ll 1$, the particle motion depends upon local gas flow conditions and upon local gas acceleration in particular. For values of λ_v/L that are neither very large nor very small, the local particle motion is dependent upon its entire history. In other words, when $\lambda_v/L \sim 1$ or $\tau_v/\tau \sim 1$, the particle has a memory of events that took place throughout a region of dimension λ_v within which significant changes in flow conditions take place, that is, the local particle motion is influenced by its entire history. Each of these three régimes presents its own peculiar problems so far as calculation is concerned.

To examine the meaning of these criteria, consider a flow field over which the characteristic velocity is u_0 and the characteristic length L . The force

to which a particle of mass m is subjected is of the order mu_0^2/L and the force exerted on the particle is $6\pi\sigma\mu u_s$, where u_s is designated the *slip velocity* of the particle with respect to the fluid. These two quantities are approximately equal, so that

$$\frac{u_s}{u_0} \approx \frac{\lambda_v}{L} \quad (2.5)$$

and consequently it follows that when $\lambda_v/L \ll 1$, the slip velocity of the particles is small in comparison with the characteristic velocity of the flow field. This fact suggests that an appropriate method for approximate calculation is to assume first that the particles and fluid move together and then to calculate deviations from this state.

On the other hand, when $\lambda_v/L \gg 1$, the slip velocity is only slightly affected by the local flow conditions. Consequently, the small interaction force between particles and fluid may be computed, in first approximation, from the local gas velocity and the velocity particles would have without interaction. The modifications thus imposed upon the particle motion become a basis for improving the interaction force estimate.

A discussion quite similar to that above holds for the thermal relaxation phenomena; the thermal range λ_T is to be compared with the characteristic length of the problem. If $\lambda_T/L \gg 1$, the particle temperature is dominated by the state at which it entered the region; if $\lambda_T/L \ll 1$, the particle temperature is dominated by local conditions. Furthermore, $\lambda_T/L \ll 1$ implies that $T_s/\Delta T \ll 1$ where T_s is the difference between particle and fluid temperature and ΔT is the characteristic temperature difference of the problem. Under conditions where $\lambda_v \approx \lambda_T$, both velocity and temperature of the particles may be calculated by similar analytical methods.

It is particularly to be noted that λ varies as the square of the particle radius, and hence within the same physical problem the conditions $\lambda^{(1)}/L \ll 1$ and $\lambda^{(2)}/L \gg 1$ may occur simultaneously.

The effect of the particulate motion upon fluid motion is, in addition to criteria mentioned, directly dependent upon the mass of particulate material in a given region. Thus, if particles of mass m number n per unit volume of space, the *particle density ratio*

$$nm/\rho_0 = \rho_p/\rho_0 \equiv \kappa \quad (2.6)$$

is a measure of the total interaction force per unit mass of fluid. If $\kappa \ll 1$ —an extreme example is a single particle moving through a fluid—the gas motion is affected very little. For $\kappa \gg 1$, the inverse is true. When, on the other hand, the particle density is such that $\kappa \sim 1$, both fluid and particle motion are affected by the interaction and a uniform method of approximation is required.

Particle-Particle Interaction

If a fluid contains particles of radii σ_1 and σ_2 , these particles will move relative to each other with velocity $u_s^{(1)} - u_s^{(2)}$ and collisions between the two-particle classes may occur. The frequency with which a particle of radius

σ_1 encounters a particle of radius σ_2 is simply $n_2\pi(\sigma_1 + \sigma_2)^2 |u_s^{(1)} - u_s^{(2)}|$. Thus a particle σ_1 will, on the average, travel a distance

$$l = \left| \frac{u_0}{u_s^{(1)} - u_s^{(2)}} \right| \frac{1}{n_2\pi(\sigma_1 + \sigma_2)^2} \quad (2.7)$$

in the time from one encounter to the next. Generally speaking, however, a particle loses its relative velocity to the fluid after travelling over its momentum range λ_v ; most of this momentum is lost early in the motion. Now the relative importance of the momentum exchange with particles σ^2 to the momentum exchange with the fluid is given by

$$\frac{\lambda_v}{l} \sim \kappa \left(\frac{\sigma u_s}{\nu} \right) \left| \frac{\sigma_1 - \sigma_2}{\sigma} \right| \quad (2.8)$$

where σ and u_s represent average particle radius and slip velocity, respectively. When $\lambda_v/l \gg 1$ the momentum exchange with fluid will have taken place before any particle encounters have occurred. When $\lambda_v/l \ll 1$ the velocity range is long and hence the particle will have transferred its momentum by collisions before significant exchange with the fluid takes place.

The mass concentration κ is at most of order unity and the slip Reynolds number, $\sigma u_s/\nu$, must be of order unity for Stokes' law to hold. For particles on opposite ends of the spectrum the factor $|(\sigma_1 - \sigma_2)/\sigma|$ may be large and collisions should therefore be accounted for. In fact this motion will appear like large spheres moving through a sort of dust cloud. For a stream of particles with uniform size, however, the collision momentum exchange is clearly negligible and that assumption will be made in the following sections.

Radiant heat transfer within the particle cloud or from particles to surface may in some cases become a significant factor governing the behavior of particles and very hot gases. When the particle concentration and size permit it, the effective "radiative conductivity" within the particle cloud may be defined as

$$k_R = l_R(4\alpha T^3)$$

where l_R is the radiative mean free path and α is Stefan's constant. The value of l_R must be small compared with significant geometric lengths of the problem for this approximation to be valid. We shall confine our attention here to those cases where it is valid. Three radiative transfer problems are of particular interest: (1) heating or cooling of particle cloud caused by radiative transfer along the flow direction; (2) radiant transfer within the particle cloud normal to the flow direction and particularly through the boundary layer; (3) radiant transfer between particles of different sizes and different temperatures.

The temperature gradient along the flow direction is of the order T/L so that the radiant transfer in this direction is $\sim \alpha T^4(l_R/L)$. The convective transport, due to particle flow along that direction is, $\rho_p u_0 c_s T$, where c_s is

the specific heat of particles. Therefore, the relative importance of the radiant transport in the flow direction is governed by the parameter

$$\left(\frac{\alpha T^4}{\rho_p u_0 c_s T} \right) \frac{l_R}{L} \quad (2.9)$$

For a high-performance rocket motor, the bracketed term is of order unity and l_R/L of the order 10^{-3} . Hence the radiative transport is negligible.

Radiant transport normal to the flow is to be compared with heat conduction in the same direction; therefore, it is the ratio of radiative to thermal conductivity that is of interest. This is

$$\frac{k_R}{k} \approx \frac{\alpha T^3 l_R}{k} \quad (2.10)$$

and under suitably high temperature conditions and appropriate particle densities, this may greatly exceed unity. Some detailed estimates of this conductivity ratio have recently been made by Penner *et al.*¹¹ for small carbon spheres based on emissivity calculations of Stull and Plass¹². The radiant transport is of especial importance in the boundary layer for extreme temperatures where the transverse temperature gradients are important; the heat transfer process may be dominated by radiation so long as l_R is much smaller than the boundary layer thickness. Under these conditions, the general effect of radiative transfer is greatly to decrease the effective Prandtl number so that the flow behaves more like a liquid metal than like a gas.

Finally, radiation heat transfer between members of the particle cloud takes place whenever, in the same region, members of the cloud have high but different temperatures. For example, if the temperature of the gas is falling along the flow direction, a large particle of radius σ_2 will have a higher temperature than a smaller particle of radius σ_1 . If T is the gas temperature, then the difference between particle temperature and gas temperature is $T_p - T \approx \Delta T (\lambda_T/L)$ where ΔT is the characteristic temperature change of the system. Then the temperatures $T_p^{(2)}$ and $T_p^{(1)}$ corresponding to particles of radius σ_2 and σ_1 satisfy the relation

$$\frac{T_p^{(2)} - T}{T_p^{(1)} - T} \approx \left(\frac{\sigma_2}{\sigma_1} \right)^2$$

It is straightforward, then, to estimate the ratio of heat radiated to particles of radius σ_1 , to that convected from these particles to the fluid. This ratio is of the order

$$\frac{k_R}{k} \left| \frac{\sigma_2 - \sigma_1}{\sigma} \right| \frac{\sigma}{l_R} \quad (2.11)$$

Since $\sigma_1/l_R \ll 1$ these two modes of heat transfer are comparable only when $k_R/k \gg 1$, that is, for reasonably large temperatures. When $(k_R/k) |(\sigma_2 - \sigma_1)/\sigma| \sigma/l_R \sim 1$ the processes that take place are of considerable interest. Heat radiates from large hot particles to small cool ones from which, in turn, it is convected to the still cooler gas. It is quite possible that

conditions of temperature, particle size, and concentration exist such that this is the controlling mechanism of cooling large particles. Then the temperature lag of all particles will approximate to that of the small ones.

The following considerations are limited in scope. They will consider, in detail, momentum exchange and heat transfer between fluid and particles ; particle collision, turbulent transport, and radiative transfer will be neglected. This in no way constitutes an assertion that these processes are unimportant but rather that, in the attempt to understand the complexities of gas-particle motion, the difficulties should be introduced gradually as an intuitive grasp of the situation is achieved and the analytic methods are developed appropriately.

3. PHYSICAL RELATIONS GOVERNING THE MOTION OF GAS CONTAINING NON-INTERACTING SOLID PARTICLES

Consider a distribution of fine, spherical particles in a perfect gas, the particulate matter being sufficiently rare that pairs or groups of particles may be considered non-interacting. At any point of space not occupied by a particle, the state of the gas is defined by its local pressure, density, temperature, and three velocity components. The state of any particle is defined by its velocity components and its temperature, both generally different from those of the neighboring gas. Moreover, within a small volume various particles may be moving in different directions and possess different temperatures depending upon their histories. Locally, the particles may be described by a distribution function $f(x_i, c_j, \vartheta, \sigma, t)$ where x_i is the position vector of the volume c_j the velocity vector of the particle, ϑ the particle temperature, and σ the particle radius which is independent of time. Then

$$f(x_i, c_j, \vartheta, \sigma, t) d^3x_i d^3c_j d\vartheta d\sigma \quad (3.1)$$

is the number of particles with radii between σ and $\sigma + d\sigma$, in a volume $x_i, x_i + dx_i$ having velocity vector in the range c_j and $c_j + dc_j$ and temperatures in the range ϑ and $\vartheta + d\vartheta$. There is no randomizing tendency in the particle states since the particles are non-interacting ; rather, there is uniformizing tendency associated with the interactions of particles with fluid. The normal state is the one where gas and particles move together ; in general, we are interested in the non-equilibrium states.

Due to the large number of particles, it is appropriate to treat the particle cloud as a continuum whenever possible. Under such circumstances, local mean quantities are calculated as averages over the distribution. The density of the particle cloud at a point is defined

$$\rho_p(x_i, t) = \int \int \int m(\sigma) f(x_i, c_j, \vartheta, \sigma, t) d^3c_j d\vartheta d\sigma \quad (3.2)$$

where $m(\sigma)$ is the mass of a particle of radius σ , and for solid homogeneous spheres is given by $\frac{4}{3}\pi\rho_s\sigma^3$ where ρ_s is the constant mass density of the solid material. In the same manner, the mean velocity components are

$$v_{pi}(x_i, t) = \frac{1}{\rho_p} \int \int \int m(\sigma) c_i f d^3c_j d\vartheta d\sigma \quad (3.3)$$

and the temperature

$$T_p(x_i, t) = \frac{1}{\rho_p} \int \int \int m(\sigma) \vartheta f d^3c_j d\vartheta d\sigma \quad (3.4)$$

To formulate equations of motion for fluid and particle cloud separately, the forces of interaction and exchange of energy must be evaluated. For low relative speeds between particles and gas stream, the force acting on a particle depends only upon its radius, the gas viscosity, and the difference between the local gas and particle velocities. However, since the gas velocity and density are variables dependent upon x_i and t , the force exerted upon the gas by relative motion of a single particle may be indicated by $\phi_k(x_i, c_j, \sigma, t)$. The force exerted upon a unit volume of gas is then

$$F_{pk}(x_i, t) = \int \int \int \phi_k(x_i, c_j, \sigma, t) f d^3c_j d\vartheta d\sigma \quad (3.5)$$

the negative of this being the force exerted by a unit volume of the gas upon the particles. Therefore, the Navier-Stokes equations take the form

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} (\tau_{ij}) + F_{pi} \quad (3.6)$$

where τ_{ij} is the conventional stress tensor for a viscous fluid. This stress tensor is related to a "smoothed" gas strain tensor in the sense that detailed gas disturbance caused by the particle motions are omitted from the gas velocity vector v_i . Since the volume occupied by the particulate matter is negligible, the continuity equation for the gas is unchanged from the usual form,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i) = 0 \quad (3.7)$$

The first law of thermodynamics, together with the equation of state, completes the description of the gas motion. In contrast with the usual situation in a gas, work is done on a gas volume by particles passing through it as well as by the neighboring gas. A single particle exerts a force $\phi_k(x_i, c_j, \sigma, t)$ on the gas and moves with a velocity $c_k - v_k$ relative to the local gas volume. Thus, it does work on the gas at the rate $(c_k - v_k)\phi_k$; the work done on the gas by the entire particulate cloud in the elementary volume is then

$$\Phi_p = \int \int \int (c_k - v_k) \phi_k(x_i, c_j, \vartheta, \sigma) f d^3c_j d\vartheta d\sigma \quad (3.8)$$

Now, furthermore, heat is transferred to the elementary gas volume from the particles contained within the element in addition to that due to thermal conduction. However, it is clear that the heat transferred to the gas from a single particle may be expressed in the form $q(x_i, c_j, \vartheta, \sigma, t)$, where the local gas state is expressed in terms of x_i, t . The total heat transfer rate to the gas by the particle cloud within the elementary volume is

$$Q_p(x_i, t) = \int \int \int q(x_i, c_j, \vartheta, \sigma, t) f d^3c_j d\vartheta d\sigma \quad (3.9)$$

The first law of thermodynamics may then be written

$$\rho \frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial x_j} + p \frac{\partial v_j}{\partial x_j} = \Phi + \frac{\partial}{\partial x_k} \left(k \frac{\partial T}{\partial x_k} \right) + \Phi_p + Q_p \quad (3.10)$$

where e is the gas internal energy, Φ the conventional viscous dissipation function, and k the thermal conductivity. The temperature T of the gas is "smoothed" to avoid local temperature variations associated with the particles.

The motions of individual particles are governed by the force of interaction between particles and gas. These individual motions are usually of no more interest in the study of observable fluid motions than are the molecular motions in a gas. Therefore, it is often appropriate to formulate the motions of the particle cloud as a continuum. This is done most directly by use of the Boltzmann equation for non-interacting particles satisfied by the distribution function

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_j} (c_j f) + \frac{\partial}{\partial c_j} \left(\frac{\phi_j f}{m} \right) + \frac{\partial}{\partial \vartheta} \left(\frac{q}{mc_s} f \right) = 0 \quad (3.11)$$

The continuum-like equations for the particle cloud follow from appropriate moments of the Boltzmann equation taken over all individual particle velocities, temperatures, and sizes. The continuity equation is obtained by multiplying Eq. (3.11) with the particle mass $m(\sigma)$, integrating over all particle velocities, temperatures, and sizes, and noting that f vanishes strongly at the upper and lower integration limits of c_j , ϑ and σ . Then utilizing the definitions (3.2) and (3.3),

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial x_j} (\rho_p v_{pj}) = 0 \quad (3.12)$$

in complete analogy with the gas-dynamical equation of continuity. Similarly, the equation of momentum conservation follows through multiplying Eq. (3.11) by the particle momentum mc_j and integrating over the appropriate space. The momentum equation for the particle cloud is then

$$\rho_p \frac{\partial v_{pi}}{\partial t} + \rho_p v_{pj} \frac{\partial v_{pi}}{\partial x_j} = -F_{pi} - \frac{\partial}{\partial x_j} (S_{ij}) \quad (3.13)$$

similar to its gas-dynamical analogue. The "slip" stress tensor

$$S_{ij} \equiv \int \int \int m(\sigma) (c_j - v_{pj}) (c_i - v_{pi}) f d^3 c_k d\vartheta d\sigma \quad (3.14)$$

arises from momentum transport due to particle motion at velocities different from the mean particle velocity. These terms are in direct correspondence with those occurring in the kinetic theory of simple gases that give rise to viscous stresses. The force component $-F_{pi}$, defined by Eq. (3.5), is the force exerted upon the particle collection by the relative motion of the fluid and particles.

Now because the temperature of a particle is not related to its translational kinetic energy in the usual kinetic theory sense, the formulation of

the first law of thermodynamics proceeds in a different manner. Multiplying Eq. (3.11) by the thermal internal energy of the particle $mc_s\vartheta$ and integrating, we obtain, with some elementary modifications

$$\rho_p c_s \frac{\partial T_p}{\partial t} + \rho_p c_s v_{pj} \frac{\partial T_p}{\partial x_j} = -Q_p - \frac{\partial}{\partial x_j} (q_{pj}) \quad (3.15)$$

where the vector

$$q_{pj} \equiv \iiint mc_s (\vartheta - T_p) (c_j - v_{pj}) f d^3c_k d\vartheta d\sigma \quad (3.16)$$

is denoted the "slip" energy flux. This term arises from various particles having different slip velocities and different temperature lags within the same region and is, roughly speaking, the thermal analogue of the slip stress tensor. The slip energy flux is of particular importance when variations in temperature lag and velocity slip are both caused by differences in particle size, for then the two terms in the integral (3.16) are correlated.

It is of some interest to investigate the kinetic energy of the particles and, since thermal and mechanical energies of the particles are not coupled directly, the kinetic energy law follows from the equations of motion. For scalar multiplication with the vector, v_{pi} yields

$$\rho_p \frac{\partial}{\partial t} (\tfrac{1}{2} v_{pi}^2) + \rho_p v_{pj} \frac{\partial}{\partial x_j} (\tfrac{1}{2} v_{pi}^2) = -v_{pi} F_{pi} - v_{pi} \frac{\partial}{\partial x_j} (S_{ij}) \quad (3.17)$$

which says simply that the rate of change of particle kinetic energy is due to (1) the work done on the particles by the fluid forces, and (2) the energy transported into the region by particle slip motions. Equations (3.15) and (3.17) may be combined to give the rate of "total particle energy" change.

4. ONE-DIMENSIONAL GAS-PARTICLE DYNAMICS

As in the conventional gas dynamics, there are many problems of gas-particle flows that can be treated as one-dimensional problems, either exactly or in good approximation. Among these are the normal shock, nozzle flow, and channel flow for both compressible and incompressible fluids. It will add greatly to the further simplifications if we assume particles of only a single size to be present in the gas. Under these restrictions, the equations of the previous section yield continuity equations for fluid and particulate phases

$$\rho u A = \dot{m} \quad (4.1)$$

$$\rho_p u_p A = \kappa \dot{m} \quad (4.2)$$

where \dot{m} is the mass flow rate of gas through a cross-section A and $\kappa \dot{m}$ is the mass flow rate of solid particles. Likewise, the momentum equations for the two phases are

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = F_p \quad (4.3)$$

$$\rho_p u_p \frac{du_p}{dx} = -F_p \quad (4.4)$$

where F_p , the force exerted by particles upon gas, is now simply

$$\begin{aligned} F_p &= n_p 6\pi\sigma\mu(u_p - u) \\ &= \rho_p a \frac{u_p - u}{\lambda_v} \end{aligned} \quad (4.5)$$

for the number density n_p of particles. Here, a denotes the local gaseous velocity of sound and

$$\lambda_v = \frac{ma}{6\pi\sigma\mu} \quad (4.6)$$

where $a/\mu = a_0/\mu_0$, a constant if the gas viscosity varies as the square root of gas temperature. It will be assumed, in the remainder of the present section, that this relationship is satisfied.

Finally, the first law of thermodynamics may be written for the gas

$$\rho u c_p \frac{dT}{dx} - u \frac{dp}{dx} = (u_p - u)F_p + Q_p \quad (4.7)$$

and for the collection of solid particles

$$\rho_p u_p c_s \frac{dT_p}{dx} = -Q_p \quad (4.8)$$

Similarly, heat transfer Q_p from particles to gas is

$$\begin{aligned} Q_p &= n_p \left(\frac{k}{\sigma} \right) 4\pi\sigma^2 (T_p - T) \\ &= \rho_p c_p a \frac{T_p - T}{\lambda_T} \end{aligned} \quad (4.9)$$

where

$$\lambda_T = \frac{3}{2} \text{Pr} \frac{ma}{6\pi\sigma\mu} \quad (4.10)$$

as before, and both a/μ and P_r are considered constant.

Together with the equation of state for the gas, the above equations provide a complete description of a one-dimensional flow problem. Sometimes it is advantageous to employ these relations in a form somewhat modified from that above. In particular, it is often convenient to treat the two phases as a single system so far as momentum conservation and the first law of thermodynamics are concerned. Clearly, from Eqs. (4.3) and (4.4)

$$\rho u \frac{du}{dx} + \rho_p u_p \frac{du}{dx} + \frac{dp}{dx} = 0 \quad (4.11)$$

from which, of course, the forces acting between phases are absent. Equations (4.7) and (4.8) may be handled similarly so that, after use of

the two momentum relations (4.3) and (4.4), one obtains a conventional energy equation

$$\rho u \left[c_p \frac{dT}{dx} + \frac{d}{dx} \left(\frac{u^2}{2} \right) \right] + \rho_p u_p \left[c_s \frac{dT_p}{dx} + \frac{d}{dx} \left(\frac{u_p^2}{2} \right) \right] = 0 \quad (4.12)$$

Now where the specific heats c_p and c_s may be considered constant, as they will here, the above equation has an elementary integral which follows from the equations of continuity for each phase. Thus

$$\left(c_p T + \frac{u^2}{2} \right) + \kappa \left(c_s T_p + \frac{u_p^2}{2} \right) = \text{const.} \quad (4.13)$$

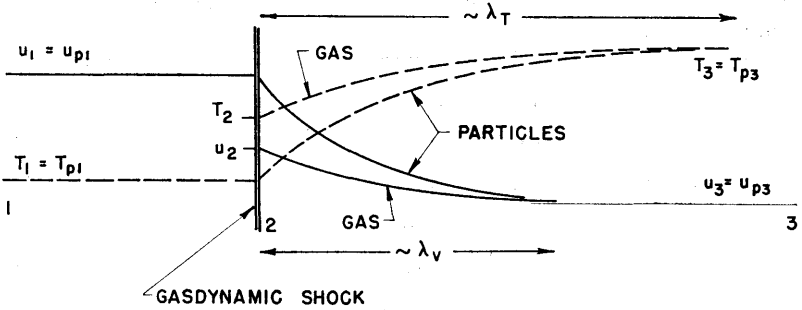


Fig. 1. Shock wave structure for gas-particle flow: schematic drawing.

where as before, κ denotes the ratio of particle mass flow to fluid mass flow. It is of interest to note here that in the special instance when the mixture velocity becomes very low, as in a settling chamber or a duct of large cross-section, then the slip velocities are negligible and $\kappa = \rho_p/\rho$.

The Normal Shock

Consider a normal shock wave in a mixture of perfect gas and a collection of small, solid, spherical particles of uniform size. So long as the solid particles are large with respect to molecular dimensions, the thickness of a gas dynamic shock is negligible in comparison with the momentum and thermal ranges of the particles. Thus, the structure of a normal shock wave in a particle-gas mixture may be thought of as a conventional gas dynamic shock followed by a relaxation zone where particles and gas come to velocity and temperature equilibrium. This problem was originally treated by Carrier¹³. By extending his analysis and calculations, this example may be used to illustrate how the fundamental parameters λ_v , λ_T , and κ influence the physical result, and to indicate the appropriate method of treatment for problems in which the particle ranges are the only significant lengths.

Denote, as indicated in Fig. 1, the conditions upstream of the shock by 1; by 2, conditions immediately downstream of the gas dynamic shock; by 3, conditions far downstream where velocity and thermal equilibrium have been reached between the two phases. The change of state in the gas

across the gas dynamic shock is given by the conventional shock relations

$$\frac{u_2}{u_1} = \frac{2}{(\gamma + 1) M_1^2} \left(1 + \frac{\gamma + 1}{2} M_1^2 \right) \quad (4.14)$$

$$\frac{T_2}{T_1} = \left(1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1) \right) \left(1 + \frac{2}{\gamma + 1} \frac{M_1^2 - 1}{M_1^2} \right) \quad (4.15)$$

The particles do not take part in the shock and their state is unchanged in the process 1-2.

On the other hand, the change of mixture state from ahead of the shock to the final equilibrium far downstream may be computed from the momentum and energy equations

$$(1 + \kappa) \rho_1 u_1^2 + p_1 = (1 + \kappa) \rho_3 u_3^2 + p_3 \quad (4.16)$$

$$\frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + \frac{u_1^2}{2} = \frac{\gamma}{\gamma - 1} \frac{p_3}{\rho_3} + \frac{u_3^2}{2} \quad (4.17)$$

where, for simplicity, we have set $c_s/c_p = 1$ and account has been taken of the equality of particle and gas temperature and velocity at 3. From these, it follows that

$$\frac{u_3}{u_1} = \frac{1}{1 + \kappa \left(\frac{2\gamma}{\gamma + 1} \right)} \left(\frac{2}{\gamma + 1} \right) \frac{1 + \left(\frac{\gamma - 1}{2} \right) M_1^2}{M_1^2} \quad (4.18)$$

so that by comparison with Eq. (4.14), the velocity ratio u_3/u_2 of the final gas speed to that just downstream of the shock becomes

$$\frac{u_3}{u_2} = \frac{1}{1 + \kappa \left(\frac{2\gamma}{\gamma + 1} \right)} \quad (4.19)$$

This result makes clear the relationship between particle effect on gas velocity and the particle density ratio κ . It is of interest to note that κ of order unity causes the gas velocity far behind the shock to be reduced to less than half its value immediately behind the shock.

The structure of the equilibration zone is found quite readily, for since the gas mass flow $\rho u = \dot{m}$ and the particle mass flow $\rho_p u_p = \kappa \dot{m}$ are constants through the zone, general momentum and energy integrals may be written

$$\dot{m} u + \kappa \dot{m} u_p + p = (1 + \kappa) \dot{m} u_3 + p_3 \quad (4.20)$$

and

$$(c_p T + \frac{1}{2} u^2) + \kappa (c_p T_p + \frac{1}{2} u_p^2) = (1 + \kappa) (c_p T_3 + \frac{1}{2} u_3^2) \quad (4.21)$$

Utilizing previous results, the differential equations for particle velocity and temperature are

$$\kappa \dot{m} \frac{du_p}{dx} = \rho_p a \frac{u - u_p}{\lambda_p} \quad (4.22)$$

$$\kappa \dot{m} \frac{dT_p}{dx} = \rho_p a \frac{T - T_p}{\lambda} \quad (4.23)$$

Together with the equation of state, Eqs. (4.20)–(4.23) describe the problem completely.

It is evident that the coordinate x does not play an essential role in the problem and that a state variable, for example u_p , may be substituted as independent variable. Then from Eqs. (4.22) and (4.23) it follows that

$$\frac{d(T_p - T_3)}{d(u_p - u_3)} = \frac{\lambda_v}{\lambda_T} \left(\frac{(T - T_3) - (T_p - T_3)}{(u - u_3) - (u_p - u_3)} \right) \quad (4.24)$$

The temperatures T and T_p of gas and particles may be expressed in terms of the gas and particle velocities using the momentum and energy integrals

$$T - T_3 = \frac{u}{R} \left[(u_3 - u) + \kappa(u_3 - u_p) \right] \quad (4.25)$$

$$T_p - T_3 = \frac{1}{\kappa c_p} \left[\frac{\gamma + 1}{2(\gamma - 1)} (u^2 - u_3^2) + \frac{\gamma}{\gamma - 1} (uu_p - u_3^2) - \frac{1}{2}(u_p^2 - u_3^2) \right] \quad (4.26)$$

and hence Eq. (4.24) may be written as a first-order non-linear differential equation for $u(u_p)$. This hodograph of the solution, obtained by numerical integration of (2.24), is shown in Fig. 2 for a particle density ratio $\kappa = 0.25$ and an upstream Mach number $M_1 = 1.6$. The broken line shows the locus of points where the gas and particles travel at the same speed. The initial point 2 corresponds to the state after the gas has undergone a normal shock and the particles have retained their state ahead of the shock. The solution to Eq. (4.12) then gives the transition between this and the final point 4 corresponding to complete equilibrium.

The geometric structure of the equilibrium zone may be found through integration of Eq. (4.22). Measured from the shock, the distance x to a point where the particle speed is u_p may be written

$$\frac{x}{\lambda_v} = \int_{u_1}^{u_p} \frac{a}{u_p} \frac{du_p}{(u - u_p)} \quad (4.27)$$

and hence x/λ_v is easily found from the hodograph solution utilizing Eq. (4.25) in calculation of the sonic speed $a(T)$. Values of x/λ_v are marked along the hodograph and the exponential approach to equilibrium is suggested by the crowding of x/λ_v values toward state 3. Values of the remaining gas and particle properties within the equilibrium zone are found algebraically from Eqs. (4.25), (4.26), and the continuity equations; these are all shown in Fig. 3, corresponding to the hodograph of Fig. 2.

Using this relatively simple problem of the normal shock wave and equilibrium zone, it is possible to observe the manner in which λ_v , λ_T , and κ each influence the structure of the solution. In the first place, changing κ from its value of 0.25 to a larger one has the effect (Eq. (4.19)) of lowering the equilibrium gas speed. If the value of κ is increased to 0.50, other conditions of the problem remaining unchanged, the hodograph is modified as indicated in Fig. 4. Roughly speaking, the hodograph is quite similar to that for $\kappa = 0.25$, but the effect is of a larger magnitude.

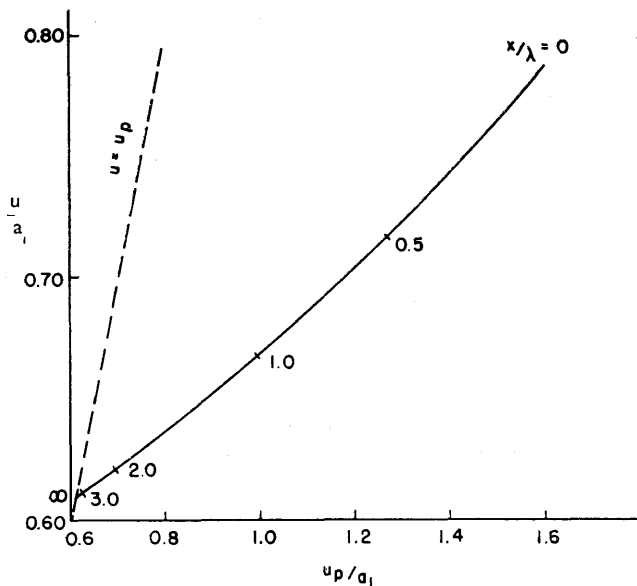


Fig. 2. Hodograph for gas-particle equilibration downstream of gasdynamic shock, $M_0 = 1.60$, $\gamma = 1.40$, $\kappa = \kappa c_s/c_p = 0.25$, $\lambda_v/\lambda_T = 1.0$.

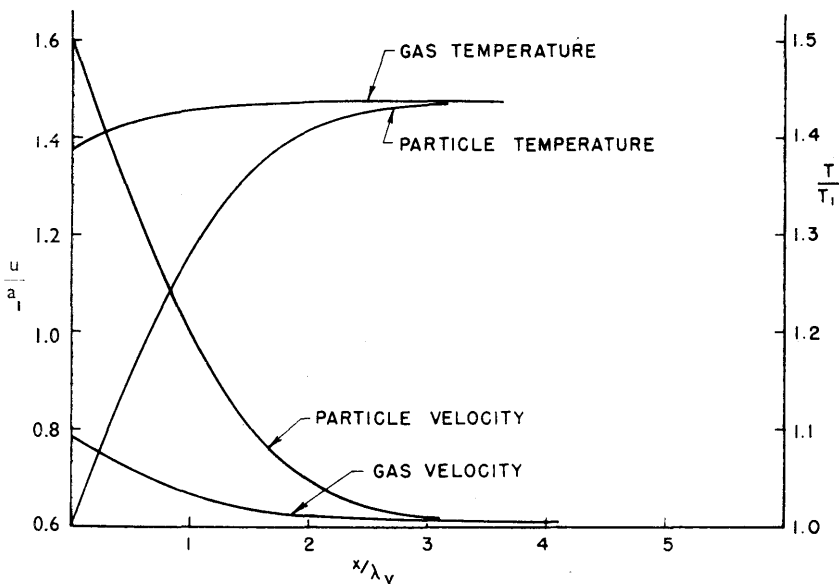


Fig. 3. Spatial equilibrium of gas-particle flow downstream of gasdynamic shock, $M_0 = 1.60$, $\gamma = 1.40$, $\kappa = \kappa c_s/c_p = 0.25$, $\lambda_v/\lambda_T = 1.0$.

When, however, the value of λ_T is increased to $10\lambda_m$, it is expected on physical grounds that the temperatures of gas and particles will approach each other less rapidly than will their velocities. This is confirmed by the third curve of Fig. 4, which shows the hodograph for $\kappa = 0.25$ and $\lambda_v/\lambda_T = 0.10$. Although the initial and end conditions are the same as those for $\kappa = 0.25$ and $\lambda_v/\lambda_T = 1.0$, the trajectory may clearly be divided into two parts. The first part consists of a relatively rapid approach toward velocity equilibrium; that is, toward the line $u = u_p$. Then follows a relatively slow region of thermal equilibration during which the gas and particle states move toward their final state, always remaining close to the line $u = u_p$. The actual process corresponds to the intuitive impression gained from the physical interpretation of the parameters λ_v , λ_T , and κ .

Flow with Small Slip

If a gas containing particles is subjected to only small accelerations and rates of temperature change, the gas and the particle cloud have nearly common velocities, common temperatures and constant density ratio, ρ_p/ρ , at all points. When these quantities are exactly equal the system behaves like a perfect gas and the problem is modified from the conventional gas dynamics only by changes in effective thermodynamic properties. For since the appropriate equations are

$$\rho u A = \dot{m} \quad (4.28)$$

$$u \frac{du}{dx} + \frac{1}{(1 + \kappa)\rho} \frac{dp}{dx} = 0 \quad (4.29)$$

$$\left(\frac{c_p + \kappa c_s}{1 + \kappa} \right) T + \frac{1}{2} u^2 = \left(\frac{c_p + \kappa c_s}{1 + \kappa} \right) T \quad (4.30)$$

$$p = \rho R T = (1 + \kappa) \rho \bar{R} T \quad (4.31)$$

the mixture behaves like a perfect gas of mass density and specific heats

$$\begin{aligned} \bar{c}_p &= \frac{c_p + \kappa c_s}{1 + \kappa} \\ \bar{c}_v &= \frac{c_v + \kappa c_s}{1 + \kappa} \\ \bar{R} &= \bar{c}_p - \bar{c}_v = \frac{R}{1 + \kappa} \end{aligned} \quad (4.32)$$

Then all of the conventional gas dynamic relationships carry over if, in place of the ratio of specific heats $\gamma = c_p/c_v$, the modified specific heat ratio

$$\bar{\gamma} = \frac{c_p + \kappa c_s}{c_v + \kappa c_s} = \frac{\gamma}{1 + \frac{\kappa c_s/c_p}{1 + \kappa c_s/c_p} (\gamma - 1)} \quad (4.33)$$

is used. For example, the speed of sound is then defined

$$a^2 = \bar{\gamma} \bar{R} T \quad (4.34)$$

$$= \frac{\bar{\gamma}}{1 + \kappa} \frac{p}{\rho}$$

and the Mach number is

$$\bar{M}^2 \equiv \frac{u^2}{a^2} \quad (4.35)$$

These clearly correspond to "frozen" sound speed and Mach number in other non-equilibrium processes.

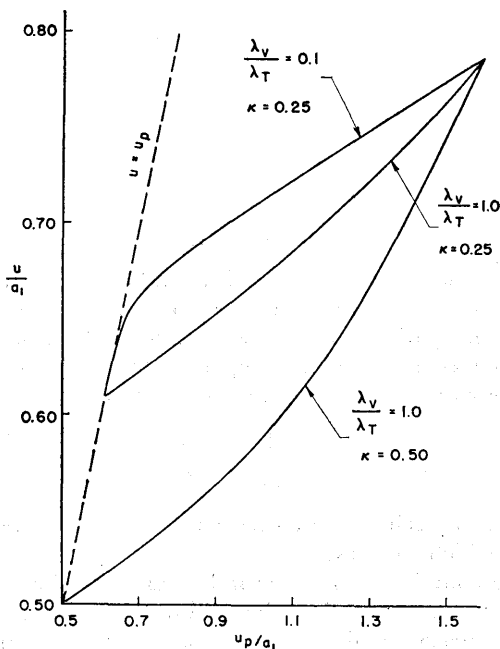


Fig. 4. Effects of particle density ratio, κ , and the ratio of momentum and thermal ranges, λ_v/λ_T , on equilibration hodograph downstream of gasdynamic shock.

Now where the slip velocity $u - u_p \equiv u_s$, and the temperature difference $T - T_p \equiv T_s$, are small in comparison with the appropriate absolute velocity and temperature values, the flow field is close to that given by the solution just described. In fact, as was indicated earlier, these variations are of the order

$$\frac{u - u_p}{u_0} \sim \frac{\lambda_v}{L} \quad (4.36)$$

$$\frac{T - T_p}{T_0} \sim \frac{\lambda_T}{L}$$

where L is the characteristic length over which a significant change in gas state occurs. Hence the criterion for small slip and small temperature difference is that $\lambda_v/L \ll 1$, $\lambda_T/L \ll 1$. When these conditions are satisfied it is appropriate to treat as dependent variables the gas state, p , ρ , T , u and the variation of particle state from the gas state

$$\begin{aligned} u - u_p &\equiv u_s \\ T - T_p &\equiv T_s \\ 1 - \frac{\rho_p}{\kappa\rho} & \end{aligned} \quad (4.37)$$

In terms of these variables the appropriate equations for the system are

$$(1 + \kappa)\rho u \frac{du}{d\xi} + \frac{dp}{d\xi} = \kappa\rho u \frac{du_s}{d\xi} \quad (4.38)$$

$$\bar{c}_p(T - T_0) + \frac{1}{2}(u^2 - u_0^2) = \frac{\kappa}{1 + \kappa} \left[c_s T_s + uu_s - \frac{1}{2}u_s^2 \right] \quad (4.39)$$

$$\rho u A = \dot{m} \quad (4.40)$$

$$p = (1 + \kappa)\rho\bar{R}T \quad (4.41)$$

where Eq. (4.38) follows from (4.11), Eq. (4.39) follows from (4.13) with the introduction of a constant reference state and a new length variable $\xi = x/L$. When dissipation is absent Eqs. (4.38)–(4.41) lead to an isentropic integral which makes the differential equation of motion redundant. In the present case, the corresponding manipulations lead to the relationship

$$\frac{T}{T_0} \left(\frac{p_0}{p} \right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} = \exp \left\{ \frac{\kappa}{1 + \kappa} \int \frac{1}{\bar{c}_p T} \left[c_s \frac{dT_s}{d\xi} + u_s \left(\frac{du}{d\xi} - \frac{du_s}{d\xi} \right) \right] d\xi \right\} \quad (4.42)$$

In the following analysis it will be convenient to take the system (4.39)–(4.42) as the one with which to work.

From suitable combinations of the individual equation for particles and for fluid it is simple to develop the additional equations of continuity, momentum and energy required for solution of the problem

$$\left(1 - \frac{\rho_p}{\kappa\rho} \right) u + u_s + \left(1 - \frac{\rho_p}{\kappa\rho} \right) u_s = 0 \quad (4.43)$$

$$u \frac{du}{d\xi} = \left(1 - \frac{\rho_p}{\kappa\rho} \right) \frac{\bar{a}u_s}{\bar{\lambda}_v/L} + u \frac{du_s}{d\xi} \quad (4.44)$$

$$u \frac{dT}{d\xi} = \left(1 - \frac{\rho_p}{\kappa\rho} \right) \left(\frac{\bar{c}_p}{c_s} \right) \frac{\bar{\lambda}_v}{\bar{\lambda}_T} \frac{\bar{a}T_s}{\bar{\lambda}_v/L} + u \frac{dT_s}{d\xi} \quad (4.45)$$

The values $\bar{\lambda}_v$ and $\bar{\lambda}_T$ differ from λ_v and λ_T in that they are based upon the frozen speed of sound. The set of Eqs. (4.39)–(4.45) is completely equivalent to the original equations and they are formulated explicitly in terms of the quantities u_s , T_s and $1 - \rho_p/\kappa\rho$.

When these three dependent variables are small, a perturbation solution naturally suggests itself, and the appropriate small quantity for the expansion is $\bar{\lambda}_v/L$ where $\bar{\lambda}_v$ and λ_T are the same order of magnitude. Then the state of the gas is written

$$\begin{aligned}\rho &= \rho^{(0)} + \frac{\bar{\lambda}_v}{L} \rho^{(1)} + \left(\frac{\bar{\lambda}_v}{L}\right)^2 \rho^{(2)} + \dots \\ u &= u^{(0)} + \frac{\bar{\lambda}_v}{L} u^{(1)} + \left(\frac{\bar{\lambda}_v}{L}\right)^2 u^{(2)} + \dots \\ p &= p^{(0)} + \frac{\bar{\lambda}_v}{L} p^{(1)} + \left(\frac{\bar{\lambda}_v}{L}\right)^2 p^{(2)} + \dots \\ T &= T^{(0)} + \frac{\bar{\lambda}_v}{L} T^{(1)} + \left(\frac{\bar{\lambda}_v}{L}\right)^2 T^{(2)} + \dots\end{aligned}\quad (4.46)$$

where each of these variables has a non-vanishing zero-th order part and all coefficients in the expansion are of order unity. The variations of the particle state from that of the gas have first order leading terms

$$\begin{aligned}u_s &= \frac{\bar{\lambda}_v}{L} u_s^{(1)} + \left(\frac{\bar{\lambda}_v}{L}\right)^2 u_s^{(2)} + \dots \\ T_s &= \frac{\bar{\lambda}_v}{L} T_s^{(1)} + \left(\frac{\bar{\lambda}_v}{L}\right)^2 T_s^{(2)} + \dots \\ 1 - \frac{\rho_p}{\kappa\rho} &= \frac{\bar{\lambda}_v}{L} \rho_s^{(1)} + \left(\frac{\bar{\lambda}_v}{L}\right)^2 \rho_s^{(2)} + \dots\end{aligned}\quad (4.47)$$

The functions giving the various order terms for each variable may be determined by substituting these expressions into Eqs. (4.39)–(4.45) and separating each equation according to powers of the small parameter $\bar{\lambda}_v/L$. For the present purpose it is sufficient to restrict consideration to the zero-th and first order terms. Equations (4.39)–(4.41) have zero-th order parts which determine completely the zero-th order solution

$$\bar{c}_p (T^{(0)} - T_0) + \frac{1}{2}(u^{(0)2} - u_0^2) = 0$$

or

$$T^{(0)} \left(1 + \frac{\bar{\gamma} - 1}{2} M^{(0)2}\right) = T_0 \left(1 + \frac{\bar{\gamma} - 1}{2} M_0^2\right) \quad (4.48)$$

$$\rho^{(0)} u^{(0)} A = \dot{m} \quad (4.49)$$

$$p^{(0)} = (1 + \kappa) \rho^{(0)} \bar{R} T^{(0)} \quad (4.50)$$

$$\frac{T^{(0)}}{T_0} = \left(\frac{p^{(0)}}{p_0}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}} \quad (4.51)$$

If the density of the mixture is denoted $\bar{\rho}^{(0)} = (1 + \kappa)\rho^{(0)}$ and it is noted that the mass flow of mixture is $(1 + \kappa)\dot{m}$, then this is clearly a conventional gas dynamic problem with modified gas properties. It represents the "isentropic" flow of a mixture where the solids always remain attached to their original gas mass.

The calculation of first order terms $\rho_s^{(1)}$, $u_s^{(1)}$ and $T_s^{(1)}$ has been greatly simplified through the manner in which Eqs. (4.43)–(4.45) have been written. For example, the right-hand side of Eq. (4.44) has a zero-th order part simply because the term $\bar{\lambda}_v/L$ occurs in the denominator of the right-hand side. Hence it follows directly that

$$u_s^{(1)} = \frac{u^{(0)}}{\bar{a}^{(0)}} \frac{du^{(0)}}{d\xi} \quad (4.52)$$

and similarly from (4.45),

$$T_s^{(1)} = \left(\frac{c_s}{\bar{c}_p} \right) \frac{\bar{\lambda}_T u^{(0)}}{\bar{\lambda}_v \bar{a}^{(0)}} \frac{dT^{(0)}}{d\xi} \quad (4.53)$$

Since the zero-th order solution is assumed known, these velocity and temperature perturbations are determined algebraically by these formulae. It is generally a convenience to rewrite the term $dT^{(0)}/d\xi$ in terms of $du^{(0)}/d\xi$ from Eq. (4.48). Utilizing this, Eq. (4.53) may be rewritten as

$$\begin{aligned} T_s^{(1)} &= -\frac{1}{\bar{c}_p} \left(\frac{c_s}{\bar{c}_p} \right) \frac{\bar{\lambda}_T}{\bar{\lambda}_v} u^{(0)} \left(\frac{u^{(0)}}{\bar{a}^{(0)}} \frac{du^{(0)}}{d\xi} \right) \\ &= -\frac{1}{\bar{c}_p} \left(\frac{c_s}{\bar{c}_p} \right) \frac{\bar{\lambda}_T}{\bar{\lambda}_v} u^{(0)} u_s^{(1)} \end{aligned} \quad (4.54)$$

In very much the same manner, it follows from Eq. (4.43) that the density perturbation is

$$\rho_s^{(1)} = -\frac{u_s^{(1)}}{u^{(0)}} = -\frac{1}{\bar{a}^{(0)}} \frac{du^{(0)}}{d\xi} \quad (4.55)$$

Evaluation of the remaining first order quantities proceeds from Eqs. (4.39)–(4.42) utilizing particularly the fact that Eqs. (4.39) and (4.42) have right-hand sides that are of the first order and are already known to that order. The perturbation quantities to be evaluated appear only on the left-hand sides and hence are given by algebraic rather than differential equations. The appropriate relations for calculating $p^{(1)}$, $T^{(1)}$, $\rho^{(1)}$ and $u^{(1)}$ are conveniently written

$$\begin{aligned} \frac{T^{(1)}}{T^{(0)}} + (\bar{\gamma} - 1) \bar{M}^{(0)2} \left(\frac{u^{(1)}}{u^{(0)}} \right) &= \frac{\kappa}{1 + \kappa} \left[\frac{c_s}{\bar{c}_p} \frac{T_s^{(1)}}{T^{(0)}} + (\bar{\gamma} - 1) \bar{M}^{(0)2} \frac{u_s^{(1)}}{u^{(0)}} \right] \\ &\equiv \frac{\kappa}{1 + \kappa} F(\xi) \end{aligned} \quad (4.56)$$

$$\frac{\rho^{(1)}}{\rho^{(0)}} + \frac{u^{(1)}}{u^{(0)}} = 0 \quad (4.57)$$

$$\frac{p^{(1)}}{p^{(0)}} = \frac{\rho^{(1)}}{\rho^{(0)}} + \frac{T^{(1)}}{T^{(0)}} \quad (4.58)$$

$$\begin{aligned} \frac{T^{(1)}}{T^{(0)}} - \frac{\bar{\gamma} - 1}{\bar{\gamma}} \left(\frac{p^{(1)}}{p^{(0)}} \right) &= \frac{\kappa}{1 + \kappa} \int_{\xi}^{\xi} \frac{1}{\bar{c}_p T^{(0)}} \left[c_s \frac{dT_s^{(1)}}{d\xi} + u_s^{(1)} \frac{du^{(0)}}{d\xi} \right] d\xi \\ &\equiv \frac{\kappa}{1 + \kappa} G(\xi) \end{aligned} \quad (4.59)$$

and the two functions $F(\xi)$ and $G(\xi)$ may be expressed in terms of the known zero-th order solution as

$$F(\xi) = (\bar{\gamma} - 1) \bar{M}^{(0)^2} \left[1 - \left(\frac{c_s}{\bar{c}_p} \right)^2 \frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right] \frac{1}{\bar{a}^{(0)}} \frac{du^{(0)}}{d\xi} \quad (4.60)$$

and

$$G(\xi) = \int_{\bar{c}_p T^{(0)}}^{\xi} \frac{1}{\bar{c}_p T^{(0)}} \left\{ \left[1 - \left(\frac{c_s}{\bar{c}_p} \right)^2 \frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right] \frac{d}{d\xi} \left(\frac{u^{(0)^2}}{\bar{a}^{(0)}} \frac{du^{(0)}}{d\xi} \right) - u^{(0)} \frac{d}{d\xi} \left(\frac{u^{(0)}}{\bar{a}^{(0)}} \frac{du^{(0)}}{d\xi} \right) \right\} d\xi \quad (4.61)$$

It is now an elementary calculation to obtain the remaining first order perturbation quantities as

$$\frac{u^{(1)}}{u^{(0)}} = - \frac{\rho^{(1)}}{\rho^{(0)}} = - \frac{\kappa}{1 + \kappa} \left(\frac{1}{1 - \bar{M}^{(0)^2}} \right) \frac{1}{\bar{\gamma} - 1} (F(\xi) - \bar{\gamma} G(\xi)) \quad (4.62)$$

$$\frac{T^{(1)}}{T^{(0)}} = \frac{\kappa}{1 + \kappa} \left(\frac{1}{1 - \bar{M}^{(0)^2}} \right) (F(\xi) - \bar{\gamma} \bar{M}^{(0)^2} G(\xi)) \quad (4.63)$$

$$\frac{p^{(1)}}{p^{(0)}} = \frac{\kappa}{1 + \kappa} \left(\frac{\bar{\gamma}}{\bar{\gamma} - 1} \right) \frac{1}{1 - \bar{M}^{(0)^2}} (F(\xi) - [1 + (\bar{\gamma} - 1) \bar{M}^{(0)^2}] G(\xi)) \quad (4.64)$$

The perturbation is not uniformly valid because of the singularity in the original problem, and it is to be expected that the perturbation quantities diverge there. The result may be made uniformly valid either by expanding the independent variable in a similar perturbation series or by utilizing as independent variable one of the thermodynamic quantities which exhibits the same singularity as the perturbation quantities. Rannie¹⁴ has chosen the latter alternative, utilizing the gas pressure as independent variable, calculating $A(p)$ later to fix the nozzle geometry.

The transformation to this independent variable is fairly straightforward if we consider the pressure, rather than the cross-sectional area, to be given as a function of ξ . It is also reasonable to assume the inverse $\xi(p)$ to be regular and single valued in the neighborhood of the singularity of Eqs. (4.62)–(4.64). Then the zero-th order problem, consisting of Eqs. (4.48), (4.50), (4.51) (but *not* the continuity equation), are considered solved for $T^{(0)}$, $\rho^{(0)}$, $u^{(0)}$ as functions of p . Then the appropriate first order perturbations are

$$\begin{aligned} u_s^{(1)} &= \frac{u^{(0)}}{\bar{a}^{(0)}} \left(\frac{du^{(0)}}{dp} \right) \left(\frac{d\xi}{dp} \right) \\ T_s^{(1)} &= - \frac{1}{\bar{c}_p} \left(\frac{c_s}{\bar{c}} \right) \frac{\bar{\lambda}_T}{\bar{\lambda}_v} \frac{u^{(0)^2}}{\bar{a}^{(0)}} \left(\frac{du^{(0)}}{dp} \right) \left(\frac{d\xi}{dp} \right) \\ \rho_s^{(1)} &= - \frac{1}{\bar{a}^{(0)}} \left(\frac{du^{(0)}}{dp} \right) \left(\frac{d\xi}{dp} \right) \end{aligned} \quad (4.65)$$

and the quantities $T^{(1)}$, $\rho^{(1)}$ and $u^{(1)}$ remain to be calculated from a set of relations analogous to (4.56), (4.58) and (4.59). These relations are

$$\begin{aligned} \frac{T^{(1)}}{T^{(0)}} + (\bar{\gamma} - 1) \bar{M}^{(0)2} \left(\frac{u^{(1)}}{u^{(0)}} \right) &= \frac{\kappa}{1 + \kappa} F(p) \\ &= \frac{\kappa}{1 + \kappa} (\bar{\gamma} - 1) \bar{M}^{(0)2} \left[1 - \left(\frac{c_s}{\bar{c}_p} \right)^2 \frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right] \frac{1}{\bar{a}^{(0)}} \left(\frac{du^{(0)}}{dp} \right) \left(\frac{d\xi}{dp} \right) \end{aligned} \quad (4.66)$$

$$\frac{\rho^{(1)}}{\rho^{(0)}} + \frac{T^{(1)}}{T^{(0)}} = 0 \quad (4.67)$$

and

$$\begin{aligned} \frac{T^{(1)}}{T^{(0)}} &= \frac{\kappa}{1 + \kappa} G(p) \\ &\equiv \frac{\kappa}{1 + \kappa} \int_p^p \frac{1}{\bar{c}_p T^{(0)}} \left\{ \left[1 - \left(\frac{c_s}{\bar{c}_p} \right)^2 \frac{\bar{\lambda}_T}{\bar{\lambda}_v} \right] \frac{d}{dp} \left(\frac{(u^{(0)2}/\bar{a}^{(0)}) (du^{(0)}/dp)}{d\xi/dp} \right) - \right. \\ &\quad \left. - u^{(0)} \frac{d}{dp} \left(\frac{(u^{(0)}/\bar{a}^{(0)}) (du^{(0)}/dp)}{d\xi/dp} \right) \right\} dp \end{aligned} \quad (4.68)$$

The solution is then simply

$$\frac{T^{(1)}}{T^{(0)}} = - \frac{\rho^{(1)}}{\rho^{(0)}} = \frac{\kappa}{1 + \kappa} G(p) \quad (4.69)$$

and

$$\frac{u^{(1)}}{u^{(0)}} = \frac{\kappa}{1 + \kappa} \left(\frac{1}{(\bar{\gamma} - 1) \bar{M}^{(0)2}} \right) (F(p) - G(p)) \quad (4.70)$$

where all expressions are clearly regular in the neighborhood of $M^{(0)} = 1$. They are, however, very inappropriate for regions where the pressure is nearly constant and $\bar{M}^{(0)}$ is sufficiently far from unity. In this circumstance the original formulation is useful.

Finally utilizing the expressions for ρ , u and the given relation $p(\xi)$, the value of $A(\xi)$ may be computed. Strictly speaking one should, at this point, consider the cross-sectional area to be expanded in the form

$$A(\xi) = A^{(0)}(\xi) + \frac{\bar{\lambda}_v}{L} A^{(1)}(\xi) + \dots \quad (4.71)$$

After substitution into the continuity relation it follows that

$$A^{(0)} = \frac{\dot{m}}{\rho^{(0)} u^{(0)}} \quad (4.72)$$

and

$$\frac{A^{(1)}}{A^{(0)}} = \frac{\kappa}{1 + \kappa} \left(G(p) + \frac{G(p) - F(p)}{(\bar{\gamma} - 1) \bar{M}^{(0)2}} \right) \quad (4.73)$$

5. PARTICLE TRAJECTORIES IN A PRANDTL-MEYER EXPANSION

In the circumstances that the particle mass ratio is very small, that is $\kappa \ll 1$, the gas dynamic flow is not altered significantly by the passage of particles through it. Then it is appropriate to calculate the forces exerted on the particle utilizing the undisturbed gas velocity. An example in which this device may be used to advantage is the calculation of trajectories for isolated particles or "test particles" in a corner expansion or Prandtl-Meyer flow.

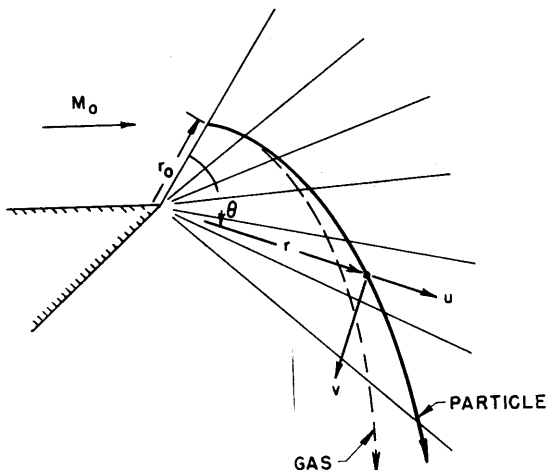


Fig. 5. Particle trajectories in a Prandtl-Meyer expansion: notation.

Consider the Prandtl-Meyer fan with initial Mach number M_0 , pressure p_0 , and temperature T_0 . The characteristics are radial lines through the corner; the state of the gas is constant along each characteristic and hence the radial velocity u , the tangential velocity v , and the thermodynamic properties depend only upon ϑ , the angle measured from the initial Mach line. The tangential velocity $v(\vartheta)$ is always the local sonic value. If r, ϑ denote the position of a particle with radius σ and mass m , Fig. 5, the conservation of radial and angular momenta may be written

$$\frac{d}{dt}(m\dot{r}) = \frac{m(r\dot{\vartheta})^2}{r} + 6\pi\sigma\mu(u - \dot{r}) \quad (5.1)$$

$$\frac{d}{dt}(mr^2\dot{\vartheta}) = 6\pi\sigma\mu(v - r\dot{\vartheta})r \quad (5.2)$$

where μ is the local viscosity of the gas, and it is assumed that the particles obey Stokes' drag law. It is convenient to introduce ϑ rather than t as the independent variable and to use the angular momentum $L = mr^2\dot{\vartheta}$ as the second dependent variable in addition to r . Then since

$$\frac{d}{dt} = \frac{L}{mr^2} \frac{d}{d\vartheta}$$

the equations of motion may be modified to the form

$$L^2 \frac{d^2}{d\vartheta^2} \left(\frac{1}{r} \right) + 6\pi\sigma\mu mr^3 v \frac{d}{d\vartheta} \left(\frac{1}{r} \right) + \frac{L^2}{r} + 6\pi\sigma\mu mr^2 u = 0 \quad (5.3)$$

$$\frac{d}{d\vartheta} (L^2) = 12\pi\sigma\mu (mr v - L) r^2 \quad (5.4)$$

The characteristic velocity of the problem is taken to be q^* , the maximum velocity to which the gas may be accelerated,

$$q^{*2} = 2c_p T_0 \left(1 + \frac{\gamma - 1}{2} M_0^2 \right) \quad (5.5)$$

Now if one introduces the velocity range based upon this characteristic velocity

$$\lambda_v = \frac{2 \rho_s q^* \sigma^2}{9 \rho_0 v_0} \quad (5.6)$$

the equations may be expressed in a convenient dimensionless form

$$\Lambda^2 \frac{d^2}{d\vartheta^2} \left(\frac{1}{R} \right) + R^3 \frac{v}{q^*} \frac{\mu}{\mu_0} \frac{d}{d\vartheta} \left(\frac{1}{R} \right) + \frac{\Lambda^2}{R} + \frac{\mu}{\mu_0} \frac{u}{q^*} R^2 = 0 \quad (5.7)$$

$$\frac{d\Lambda^2}{d\vartheta} = 2R^2 \frac{\mu}{\mu_0} \left(R \frac{v}{q^*} - \beta\Lambda \right) \quad (5.8)$$

The dimensionless dependent variables are

$$R = \frac{r}{\beta^2 \lambda_v} \quad \Lambda = \frac{L}{mq^* \beta^3 \lambda_v} \quad (5.9)$$

where $\beta^2 \equiv (\gamma - 1)/(\gamma + 1)$ and the quantities u/q^* , v/q^* , μ/μ_0 depend upon the flow field and hence are functions of ϑ only.

It will be assumed that the particles approach the initial characteristic with the free stream velocity and a given displacement from the plane surface. Thus one knows the initial values $r_{(0)}$, $dr/d\vartheta_{(0)}$ and $L_{(0)}$. In terms of the stream values, take

$$\begin{aligned} r(0) &= r_0 \\ \frac{dr}{d\vartheta}(0) &= \frac{dr}{dt}(0) \left/ \frac{d\vartheta}{dt}(0) \right. = r_0 \sqrt{(M_0^2 - 1)} \\ L(0) &= mr_0 v_0 \end{aligned} \quad (5.10)$$

Then the dimensionless dependent variables have the initial values

$$R(0) = \frac{r_0}{\beta^2 \lambda_v} \quad (5.11)$$

$$\frac{dR}{d\vartheta}(0) = R(0) \sqrt{(M_0^2 - 1)} \quad (5.12)$$

$$\Lambda(0) = \frac{R(0)}{\sqrt{\frac{2}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} M_0^2 \right)}} \quad (5.13)$$

To complete the formulation of the problem we require only the gas velocities and viscosity ratio expressed in terms of the angle variable ϑ . These may be written down most conveniently in the form

$$\frac{u}{q^*} = \sin [\beta \vartheta + \alpha] \quad (5.14)$$

$$\frac{v}{q^*} = \beta \cos [\beta \vartheta + \alpha] \quad (5.15)$$

$$\frac{\mu}{\mu_0} = \left(\frac{T}{T_0} \right)^\epsilon = [1 + \beta^2 (M_0^2 - 1)]^\epsilon \cos^{2\epsilon} (\beta \vartheta + \alpha) \quad (5.16)$$

where the auxiliary angle α is defined through

$$\sec \alpha = \sqrt{\left[\frac{2}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} M_0^2 \right) \right]} \quad (5.17)$$

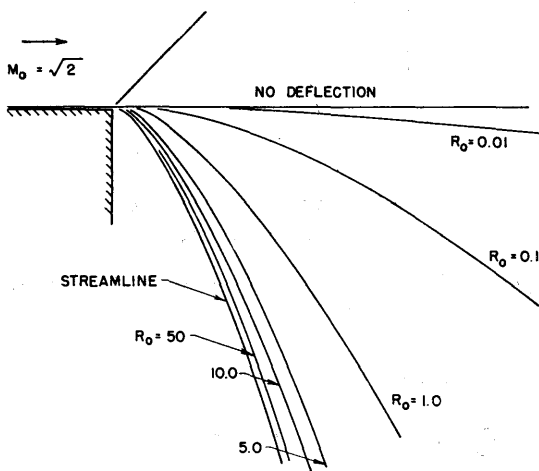


Fig. 6. Trajectories for particles of various reduced initial radii. $M_0 = \sqrt{2}$, $\gamma = 1.40$, $\mu \sim T$.

Explicitly, then, the differential equations of particle motion may be written

$$\Lambda^2 \frac{d^2}{d\vartheta^2} \left(\frac{1}{R} \right) + \beta \sec^{2\epsilon}(\alpha) \cos^{2\epsilon+1}(\beta \vartheta + \alpha) R^3 \frac{d}{d\vartheta} \left(\frac{1}{R} \right) + \frac{\Lambda^2}{R} + \sec^{2\epsilon}(\alpha) \sin(\beta \vartheta + \alpha) \cos^{2\epsilon}(\beta \vartheta + \alpha) R^2 = 0 \quad (5.18)$$

$$\frac{d}{d\vartheta} (\Lambda^2) = 2\beta \sec^{2\epsilon}(\alpha) \cos^{2\epsilon}(\beta \vartheta + \alpha) R^2 [R \cos(\beta \vartheta + \alpha) - \Lambda] \quad (5.19)$$

Particle trajectories may be calculated numerically utilizing Eqs. (5.18) and (5.19) above and the initial conditions (5.11), (5.12), (5.13). It is particularly to be noted that for a given initial Mach number only the initial

value of R_0 need be specified; that is, all variations of particle size, initial distance from wall, and gas properties are accounted for through the dimensionless parameters employed. Some typical particle trajectories are shown in Fig. 6 for an initial stream Mach number $M_0 = \sqrt{2}$. For large R_0 ,

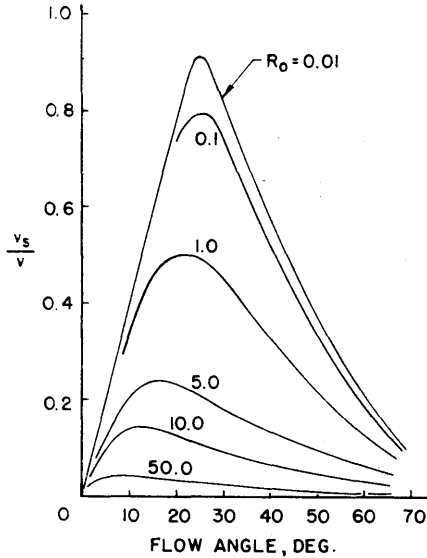


Fig. 7. Tangential slip velocity of particles in Prandtl-Meyer flow. $M_0 = \sqrt{2}$, $\gamma = 1.40$, $\mu \sim T$.

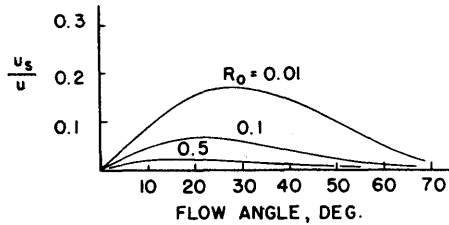


Fig. 8. Radial slip velocity of particles in Prandtl-Meyer flow. $M_0 = \sqrt{2}$, $\gamma = 1.40$, $\mu \sim T$.

that is, for $\lambda_v \ll r_0$, the particle slip is small as we intuitively expect. Conversely, for small R_0 , that is, where $\lambda_v \gg r_0$, the particle trajectory shows only small deflection from its initial state. The radial and tangential slip velocities, shown in Figs. 7 and 8, respectively, demonstrate that the departure of particle trajectories from streamlines is due more to tangential than radial slip. The appropriate picture is not that the particles are "thrown centrifugally" through the gas flow. Rather a particle moves with very nearly the local radial gas speed, spending a protracted time within each angular

increment of the flow and hence moves through a larger radial distance by the time it has transversed a given angle.

The effects of specific heat ratio and viscosity temperature dependence are shown in Fig. 9 where tangential slip velocity for a particle of $R_0 = 1.0$ is given for $\gamma = 1.2$ and for $\mu \sim T^{1/2}$ to be compared with the results for $\gamma = 1.4$ and $\mu \sim T$ given previously. The viscosity variation, which one might expect to have substantial influence at large expansion ratios, fails to affect the results by a significant amount. The specific heat ratio does have the considerable influence indicated and it is of interest to note that most of the effect shown comes about through the variation of

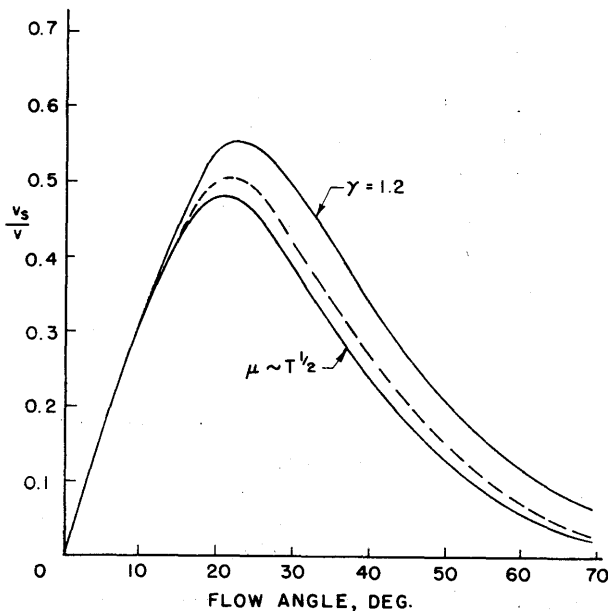


Fig. 9. Effects of specific heat ratio and viscosity-temperature dependence upon particle trajectories in Prandtl-Meyer flow. $R_0 = 1.0$, $M_0 = \sqrt{2}$.

$\beta \equiv \sqrt{[(\gamma - 1)/(\gamma + 1)]}$ in the parameters R and Λ . Thus the effect of varying specific heat ratio may be crudely accounted for with only a single calculation of $R(\vartheta)$ and $\Lambda(\vartheta)$.

For small particles and high slip velocities the Stokes drag law should be modified to account for deviations from "creeping flow" and for the effects of molecular slip flow. In the range appropriate to our problems the slip is determined by whether or not M_p/R_e , the ratio of the particle Mach number to Reynolds number, is small in comparison to unity. It is not unusual for this ratio to be significant for very small particles and Rannie¹⁴ has proposed an empirical drag law obtained by multiplying the Stokes drag by a factor containing additive Reynolds number and slip corrections in the form $6\pi\sigma\mu[F(R_e) + 4.05\sqrt{\gamma}(M_p/R_e)]^{-1}$. To show the importance of slip in a particular instance, we suppress (unjustifiably) the Reynolds number dependence and take the drag law with $F(R_e) = 1$. This modification has

the analytical effect of multiplying the last term in Eqs. (5.18) and (5.19) by the factor

$$\left[1 + 4.05\sqrt{\gamma} \frac{M_p}{R_e} \right]^{-1} \quad (5.20)$$

The results for $M_0 = 3.0$ and other conditions in the range appropriate to rocket nozzle exhaust give the curves of Fig. 10. Results are shown for

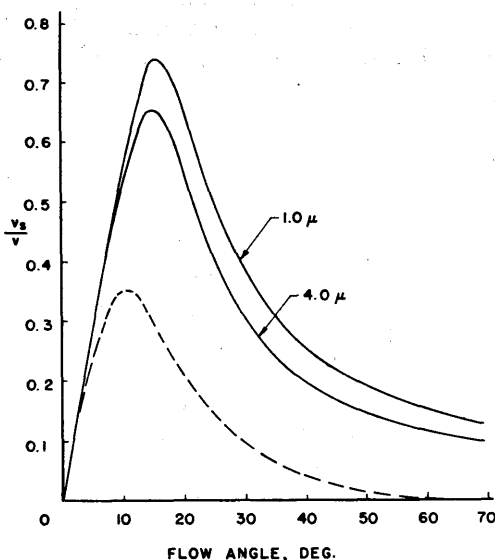


Fig. 10. Effects of molecular slip flow and Reynolds number corrections to tangential slip velocity of particles in Prandtl-Meyer flow. $M_0 = 3.0$, $\gamma = 1.25$, $\mu \sim T^{\frac{1}{2}}$, $a_0 = 3,100$ ft/sec.

1.0μ and 4.0μ particles and clearly emphasize the importance of this phenomenon. Truly accurate calculations must await better information on slip flow drag measurements but the present results certainly indicate the order of magnitude of the modification that can be expected. The effect seems to be of greatest importance for values of $R_0 = 1.0$. For smaller R_0 (larger particles) the molecular slip is less important and for larger R_0 (smaller particles) the particle follows the gas stream so closely that the change is of little absolute interest.

Finally it should be noted that when particle Reynolds number is considerably larger than unity and when the particle Mach number correction is of importance, the particle trajectories form a three-parameter family rather than the one-parameter family presented here.

6. LAMINAR BOUNDARY LAYER FOR INCOMPRESSIBLE FLUID-PARTICLE SYSTEMS

The study of the boundary layer in fluid-particle systems is of special interest because of the influence of the particles upon the wall shear and heat transfer coefficients, the possible tendency of particles to collect near a wall,

and the problem of particle impingement on the wall. In order to achieve some basic understanding of the boundary layer problem, it seems well to look first at some of the simplest laminar boundary layer examples and to see in what important ways they are modified from those without the particulate matter. So far as turbulent boundary layer is concerned, it appears that any real understanding of the problem will be achieved only as the result of a substantial experimental program and the development of appropriate experimental techniques. This is hardly surprising, since the situation was the same for turbulent boundary layers without particles.

Two questions arise immediately when the laminar boundary layer is considered: (1) is the boundary layer approximation satisfied for both fluid and particle flows, and (2) is it necessary to account for the lateral force on the small particles due to the fact that they are moving through a shearing flow? Without pursuing the subject in detail, it may be stated that when κ is at most of order unity the boundary layer approximation for the fluid is valid under conditions where it would be valid without particles. Whether a similar approximation is valid for the particle flow depends upon the value of λ_v and the geometry of the problem. For example, the boundary layer approximation for particles is completely valid for the Blasius problem but invalid for a stagnation point flow for a distance of several λ_v about the stagnation point. Concerning the lateral forces on the particles we do not have a completely sound basis to account for them. Within the first order Stokes approximation the force vanishes while the problem has not been treated completely satisfactorily in the Oseen approximation. This lateral force will be neglected in the following analysis.

The boundary layer formed by an incompressible fluid moving over a plane surface is described by the usual boundary layer equations for the fluid

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6.1)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + F_{px} \quad (6.2)$$

where the momentum equation is augmented by the particle force in the x -direction. The equations of motion for the particles may also be written down simply when we permit only particles of a single fixed geometric size. The continuity relation is

$$\frac{\partial}{\partial x} (\rho_p u_p) + \frac{\partial}{\partial y} (\rho_p v_p) = 0 \quad (6.3)$$

while the equations of motion parallel and normal to the surface are

$$\rho_p u_p \frac{\partial u_p}{\partial x} + \rho_p v_p \frac{\partial u_p}{\partial y} = -F_{px} \quad (6.4)$$

$$\rho_p u_p \frac{\partial v_p}{\partial x} + \rho_p v_p \frac{\partial v_p}{\partial y} = -F_{py} \quad (6.5)$$

The two forces F_{px} and F_{py} exerted by the particles on the fluid may be written, according to Stokes law,

$$F_{px} = \rho_p u_0 \frac{u_p - u}{\lambda_v} \quad (6.6)$$

$$F_{py} = \rho_p u_0 \frac{v_p - v}{\lambda_v} \quad (6.7)$$

where λ_v is the momentum range $(mu_0)/(6\pi\sigma\mu)$ based upon the undisturbed free stream velocity.

Now since the particles are not constrained by an equation of state, the continuity Eq. (6.3) constitutes a very weak condition on their motion normal to the plate. This motion is governed by the normal component of

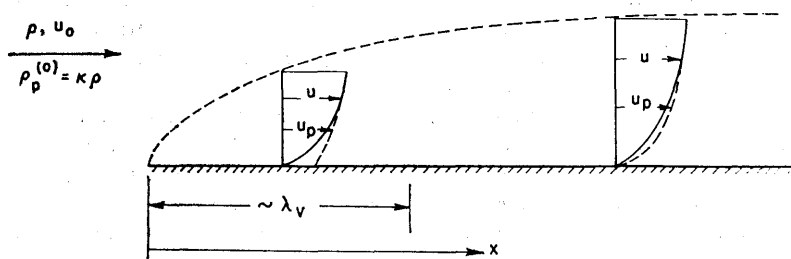


Fig. 11. Schematic drawing of gas-particle boundary layer flow showing characteristic regions near and far from leading edge.

Stokes drag, Eq. (6.7), and the initial condition on the particles. Since the particles in the free stream move parallel to the surface, the motion of particles normal to the surface is governed by the normal motion of the fluid, through the Stokes drag law. It is not difficult to show that the particle slip velocity, $v_p - v$, normal to the plate is of the same order as the normal pressure gradient within the boundary layer. This quantity will be neglected in the following calculations and it will be approximated that

$$v_p \cong v \quad (6.8)$$

It is quite clear intuitively that at least two separate fluid-particle flow régimes exist. These are indicated in Fig. 11. Very close to the leading edge, the fluid velocity close to the wall drops immediately to zero and the particles initially transported with this fluid require an axial length of several λ_v before they are retarded to near fluid velocities. Hence a region $\sim \lambda_v$ about the leading edge constitutes the régime of high slip. Farther back on the plate at distances large with respect to the momentum range, the particles have assumed nearly fluid velocity. The only factor which prevents them from actually attaining fluid velocity is the fluid retardation along particle paths due to the thickening of the boundary layer. One recognizes then that the distance from the leading edge is the characteristic length

with which to compare λ_v ; thus when $\lambda_v/x \gg 1$ the particle motion is governed by its initial state, when $\lambda_v/x \ll 1$ the particles exhibit only a small slip with respect to the fluid. In the transition region $\lambda_v/x \sim 1$ both initial conditions and local boundary layer structure are of importance. We shall investigate here only the case $\lambda_v/x \ll 1$.

The classical Blasius problem concerns itself with the uniform free stream flow parallel to a semi-infinite flat plate whose leading edge is located at $x = y = 0$. Then within the boundary layer approximation, the pressure variation along the plate vanishes in Eq. (6.2).

To carry out the solution of this problem it is convenient to introduce variables of a convenient physical magnitude. The fluid velocities u and v are appropriate variables and, together with the fact that $v_p = v$ it remains only to account for the particle velocity and the particle density ρ_p . Define as new dependent variables the slip velocity ratio

$$\frac{u_p - u}{u_0}(x, y) \quad (6.9)$$

and the particle density variation

$$\frac{\rho_p}{\rho_p^{(0)}} \quad (6.10)$$

where $\rho_p^{(0)}$ is the particle mass density of the free stream and $\rho_p^{(0)}/\rho \equiv \kappa$. The differential equations may be rewritten to conform to this choice of variables. The continuity equation for the fluid stands in its present form, Eq. (6.1). The continuity equation for the particle continuum may be modified, through use of the fluid continuity, to read

$$\begin{aligned} \frac{\rho_p}{\rho_p^{(0)}} \frac{\partial}{\partial x} \left(\frac{u_p - u}{u_0} \right) + \frac{u_p - u}{u_0} \frac{\partial}{\partial x} \left(\frac{\rho_p}{\rho_p^{(0)}} \right) + \\ + \frac{u}{u_0} \frac{\partial}{\partial x} \left(\frac{\rho_p}{\rho_p^{(0)}} \right) + \frac{v}{u_0} \frac{\partial}{\partial y} \left(\frac{\rho_p}{\rho_p^{(0)}} \right) = 0 \end{aligned} \quad (6.11)$$

It is convenient also to obtain a "force free" equation for the medium, that is, one that treats the fluid and particles as a single continuum. The desired result is obtained directly as the sum of Eqs. (6.2) and (6.4). After some reduction, the resulting equation may be put in a form utilizing the new dependent variables

$$\begin{aligned} \left[1 + \kappa \left(\frac{\rho_p}{\rho_p^{(0)}} \right) \right] \left[\frac{u}{u_0} \frac{\partial u}{\partial x} + \frac{v}{u_0} \frac{\partial u}{\partial y} \right] - v \frac{\partial^2}{\partial y^2} \left(\frac{u}{u_0} \right) + \\ + \kappa \left(\frac{\rho_p}{\rho_p^{(0)}} \right) \left[u \frac{\partial}{\partial x} \left(\frac{u_p - u}{u_0} \right) + v \frac{\partial}{\partial y} \left(\frac{u_p - u}{u_0} \right) + \left(\frac{u_p - u}{u_0} \right) \frac{\partial u}{\partial x} \right] + \\ + \kappa u_0 \left(\frac{\rho_p}{\rho_p^{(0)}} \right) \left(\frac{u_p - u}{u_0} \right) \frac{\partial}{\partial x} \left(\frac{u_p - u}{u_0} \right) = 0 \end{aligned} \quad (6.12)$$

To complete the formulation of the problem, a second combination of

particles and fluid momentum equations may be used. In order to obtain the appropriate expressions in terms of $(u_p - u)/u_0$, etc., it is convenient to subtract $1/\rho_p$ times Eq. (6.5) from $1/\rho$ times Eq. (6.2), substituting from Eq. (6.6) for the mutual force term where appropriate. Again, after a little reduction

$$u \frac{\partial}{\partial x} \left(\frac{u_p - u}{u_0} \right) + v \left(\frac{\partial y u_p - u}{\partial u_0} \right) + (u_p - u) \frac{\partial}{\partial x} \left(\frac{u}{u_0} \right) + (u_p - u) \frac{\partial}{\partial x} \left(\frac{u_p - u}{u_0} \right) + \nu \frac{\partial^2}{\partial y^2} \left(\frac{u}{u_0} \right) = - \left(1 + \kappa \frac{\rho_p}{\rho_p^{(0)}} \right) \frac{u_0}{\lambda_v} \left(\frac{u_p - u}{u_0} \right) \quad (6.13)$$

Now a very important feature may be observed from Eq. (6.12). The terms have been grouped according to powers of the small quantity $(u_p - u)/u_0$; the first group is independent of it, the second group is linear, the third quadratic. When the slip velocity vanishes, the remaining terms are simply those associated with the conventional Blasius layer except that the momentum terms are multiplied by a factor $1 + \kappa$. Thus in the approximation of negligible slip, the boundary layer acts like one with a kinematic viscosity

$$\nu^* = \frac{1}{1 + \kappa} \nu \quad (6.14)$$

In reality, this is simply because the particles contribute to the mixture density but do not contribute to the viscosity. In choosing the appropriate independent variable for the problem, we are led to the choice x and η where now we define

$$\eta = \frac{y}{\sqrt{[\nu^* x]/u_0}} = \frac{\sqrt{(1 + \kappa)} y}{\sqrt{[\nu x/u_0]}} \quad (6.15)$$

With finite slip this boundary layer has no simple similarity in the sense of the usual Blasius boundary layer, physically, because the nature of the slip depends on the parameter λ_v/x . Omitting details, the dependent variables will be chosen from the stream function $\psi(x, \eta)$, the particle slip ratio $(u_p - u)/u_0(x, \eta)$ and the density ratio $(\rho_p)/\rho_p^{(0)}(x, \eta)$ and it proves proper to express each of them in power series of the slip parameter λ_v/x .

$$\psi(x, \eta) = \sqrt{\nu^* u_0 x} \left\{ f^{(0)}(\eta) + \frac{\lambda_v}{x} f^{(1)}(\eta) + \left(\frac{\lambda_v}{x} \right)^2 f^{(2)}(\eta) + \dots \right\} \quad (6.16)$$

$$\frac{u_p - u}{u_0} = \frac{\lambda_v}{x} g^{(1)}(\eta) + \left(\frac{\lambda_v}{x} \right)^2 g^{(2)}(\eta) + \dots \quad (6.17)$$

$$\frac{\rho_p}{\rho_p^{(0)}} = 1 + \frac{\lambda_v}{x} h^{(1)}(\eta) + \left(\frac{\lambda_v}{x} \right)^2 h^{(2)}(\eta) + \dots \quad (6.18)$$

The differential equations (6.11), (6.12) and (6.13) may be transformed to the new dependent and independent variables including the power series

in λ_v/x . Thus it is possible to decompose each of the differential equations according to the algebraic order of the multiplicative factor $(\lambda_v/x)^n$. We shall be concerned only with the zero-th and first orders in the following discussion.

The zero-th order part Eq. (6.12) gives the initial term in the stream function and the differential equation for $f^{(0)}(\eta)$ is

$$\frac{d^3 f^{(0)}}{d\eta^3} + \frac{1}{2} f^{(0)} \frac{d^2 f^{(0)}}{d\eta^2} = 0 \quad (6.19)$$

Thus in this approximation the flow corresponds to a Blasius distribution but in terms of a vertical scale modified to account for the particle density ratio. This result is quite reasonable inasmuch as the zero-th order terms really consider the particles and fluid to move together.

The first order term in the slip velocity expansion follows from the zero-th order term in Eq. (6.13); this is due, of course, to the λ_v^{-1} that multiplies the right-hand side and reduces its order by one. The result is simply

$$g^{(1)}(\eta) = \frac{1}{2} f^{(0)} \frac{d^2 f^{(0)}}{d\eta^2} \quad (6.20)$$

Physically, this result confirms our earlier statement that the particle slip velocity is proportional to the local acceleration, for the term

$$\frac{1}{2} f^{(0)} \frac{d^2 f^{(0)}}{d\eta^2}$$

corresponds to the local acceleration of the zero-th order flow.

From the particle continuity equation (6.11), we obtain the first order term for the particle density variation.

$$\begin{aligned} \frac{1}{2} f^{(0)} \frac{dh^{(1)}}{d\eta} + \frac{df^{(0)}}{d\eta} h^{(1)} &= -g^{(1)} - \frac{\eta}{2} \frac{dg^{(1)}}{d\eta} \\ &= -\left[f^{(0)} \left(1 - \frac{\eta}{4} f^{(0)} \right) + \frac{\eta}{2} \frac{df^{(0)}}{d\eta} \right] \frac{1}{2} \frac{d^2 f^{(0)}}{d\eta^2} \end{aligned} \quad (6.21)$$

where Eqs. (6.19) and (6.20) have been used in the reduction of the right-hand side. Finally, the first order modification $f^{(1)}(\eta)$ to the stream function follows from the first order term in Eq. (6.12)

$$\begin{aligned} \frac{d^3 f^{(1)}}{d\eta^3} + \frac{1}{2} f^{(0)} \frac{d^2 f^{(1)}}{d\eta^2} + \frac{df^{(0)}}{d\eta} \frac{df^{(1)}}{d\eta} - \frac{1}{2} \frac{d^2 f^{(0)}}{d\eta^2} f^{(1)} \\ = -\frac{\kappa}{1+\kappa} \left(\frac{1}{2} f^{(0)} \frac{d^2 f^{(0)}}{d\eta^2} \right) \left[h^{(1)} + \frac{3}{2} \frac{df^{(0)}}{d\eta} + \frac{\eta}{2} \frac{d^2 f^{(0)}}{d\eta^2} - \frac{1}{4} f^{(0)2} \right] \end{aligned} \quad (6.22)$$

It is important to notice here at what point the particle parameters enter the first order problem. First, the particle density ratio κ enters the definition of the vertical similarity η , secondly λ_v enters in the expansion process as the characteristic length along the plate, and finally the particle density ratio κ enters as a multiplicative factor in the homogeneous term of (6.22). It follows immediately that $f^{(0)}(\eta)$, $g^{(1)}(\eta)$, $h^{(1)}(\eta)$

and $(1 + \kappa/\kappa)f^{(1)}(\eta)$ are universal functions of η and, once calculated, provide the first order solution for this fluid-particle boundary layer regardless of λ_v or κ .

Detailed calculations of these functions have been made and the results are shown in Figs. 12, 13 and 14. Of particular interest is the shear law

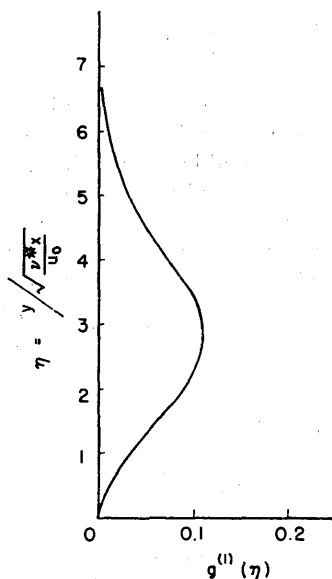


Fig. 12. Coefficient of first-order perturbation term for particle velocity: λ_v/κ small.

which results from the particle alterations to the boundary layer. Formulating the shear coefficient in the usual manner

$$\frac{\tau}{\rho u_0^2} = \frac{0.332}{\sqrt{R_x}} \sqrt{(1 + \kappa)} \left(1 + 0.49 \frac{\lambda_v}{\kappa} \frac{\kappa}{1 + \kappa} \right) \quad (6.23)$$

where we recognize $0.332/\sqrt{R_x}$ as the shear coefficient for the fluid boundary layer without particles. The factor $\sqrt{(1 + \kappa)}$ multiplying the usual shear coefficient gives the result for no particle slip and represents a minimum value for the shearing stress. The first order correction $0.49(\lambda_v/\kappa)(\kappa/1 + \kappa)$ gives the reduction in shear stress due to particle slip reduction along the flow path.

It is clear that the heat transfer through the incompressible boundary layer may be treated by the same technique used to calculate the momentum boundary layer and shear stress. This calculation is particularly simple when the fluid has a Prandtl number of unity and when the thermal and momentum ranges of particles are equal. For then, the two problems are strictly similar and the result is just the heat transfer coefficient for fluid

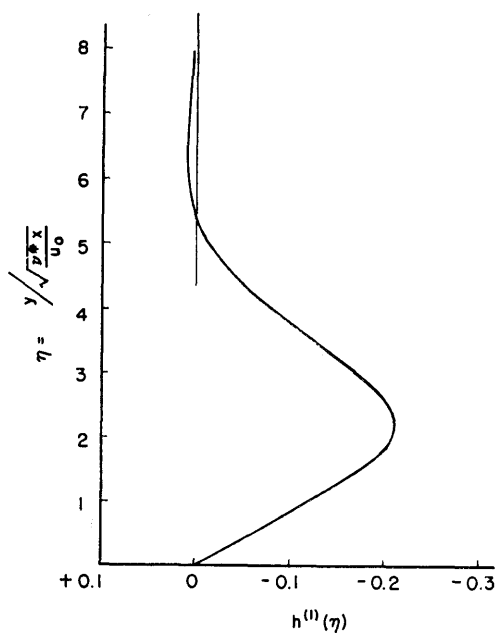


Fig. 13. Coefficient of first-order perturbation term for particle density ratio: λ_v/κ small.

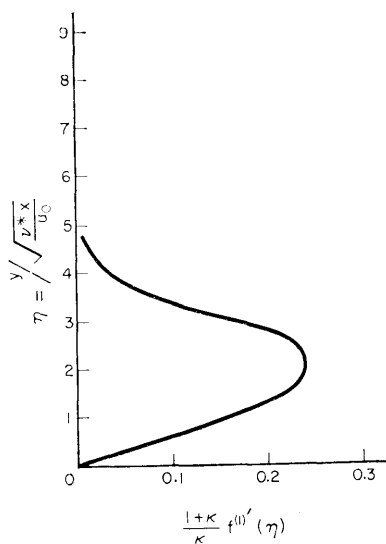


Fig. 14. Coefficient of first-order perturbation term for fluid velocity: λ_v/κ small.

alone multiplied by $\sqrt{(1 + \kappa)(1 + 0.49(\lambda_v/\kappa)(\kappa/1 + \kappa))}$, the same factor appearing in Eq. (6.23) for the shear coefficient.

7. CONCLUDING REMARKS

It has been the two-fold aim of this paper (1) to define and show the significance of the several novel parameters or similarity groups that enter the dynamics of gas-particle flow systems, and (2) to indicate some of the fields of classical fluid mechanics into which gas-particle flow theory may be extended. It is clear that only the surface has been touched, for example, in treating two- and three-dimensional flow fields with particulate matter. When $\lambda_v/L \ll 1$, the analysis proceeds in much the same way as that for chemically near-equilibrium flow systems. For perturbation theory, the analysis may be carried out by usual methods regardless of the magnitude of λ_v/L . Considerable extension can be effected in this direction by exploiting methods already at hand. When $\lambda_v/L \gg 1$ it is the solid boundaries which are a major factor in determining the particle flow pattern and the problem takes on some aspects of rarified gas dynamics.

It is equally clear that a wide variety of boundary-layer problems may be treated along the lines indicated in this paper. There is no essential difficulty in treating the compressible boundary layer. One problem which emphasizes the contrast between boundary layer flows with and without particles is the laminar boundary layer on a curved wall. Here, although the effect of curvature on the strictly fluid layer is negligible when the radius of curvature is large compared with the boundary layer thickness, the effect of curvature on the particles is indeed a first-order one and imposes a corresponding modification on the fluid velocity distribution.

One of the problems that must be faced in the course of developing this field is a rational study of particle-particle collisions. This is important for particle-size spectra with great size spread. The collisions tend to impart a degree of randomness to the particle motion and promote diffusion phenomena; the fluid, however, tends to inhibit this randomness and to impart an orderly motion governed by the fluid. Moreover, the process of particle-particle interaction is not yet satisfactorily clear due to the mutual interaction of the flow fields of colliding particles.

In general, the author believes the field of gas-particle flows to be one in which fluid mechanicians may profitably utilize their techniques and experience in the fundamental development of an interesting field and, perhaps, of an important technology.

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